# WHAT TO EXPECT WHEN YOU'RE EXPECTING

## SOLVING LINEAR RATIONAL EXPECTATIONS MODELS

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# STARTING POINT

- At this point, we've linearized our model and we've cast the equations of the model *n* equations into the following form

$$\mathbf{0}_{n\times 1} = \mathbf{A}\mathbb{E}_t\left[x_{t+1}\right] + \mathbf{B}x_t + \mathbf{C}x_{t-1} + \mathbf{F}u_t \tag{1}$$

- $x_t$ is an $n \times 1$ vector of the variables in our model and $u_t$ is an $m \times 1$ vector of serially-uncorrelated exogenous shocks such that $\mathbb{E}_t\left[u_{t+1}\right] = \mathbf{0}_{m\times 1}$
- **The Goal**: Find a law of motion for $x_t$ that *satisfies this equation*, under the assumption that *expectations are formed using this law of motion*

# UNDETERMINED COEFFICIENTS

- We conjecture a solution of the following form

$$x_t = \mathbf{P}x_{t-1} + \mathbf{R}u_t \tag{2}$$

  and plug it into equation (1)

$$\mathbf{0}_{n\times 1} = \mathbf{A}\mathbb{E}_t\left[\mathbf{P}(\mathbf{P}x_{t-1} + \mathbf{R}u_t) + \mathbf{R}u_{t+1}\right] + \mathbf{B}x_t + \mathbf{C}x_{t-1} + \mathbf{F}u_t$$
$$\Rightarrow \mathbf{B}x_t = -\left(\mathbf{AP}^2 + \mathbf{C}\right)x_{t-1} - \left(\mathbf{APR} + \mathbf{F}\right)u_t$$

- Further, by our conjecture

$$\mathbf{B}x_t = \mathbf{B}\left(\mathbf{P}x_{t-1} + \mathbf{R}u_t\right)$$

# UNDETERMINED COEFFICIENTS

- Now we know that, in order for the conjectured equation to be correct, two equations must hold:

$$\mathbf{0}_{n\times n} = \mathbf{A}\mathbf{P}^2 + \mathbf{B}\mathbf{P} + \mathbf{C} \tag{3}$$

$$\mathbf{R} = -\left(\mathbf{A}\mathbf{P} + \mathbf{B}\right)^{-1}\mathbf{F} \tag{4}$$

- Solving the problem is now based on solving equation (3), which is just a matrix algebra problem!

# SOLVING THE MATRIX QUADRATIC

- Solving the problem is based, in part, on noticing that equation (3) can be written as

$$\Phi_0 \begin{bmatrix} \mathbf{I}_n \\ \mathbf{P} \end{bmatrix} = \Phi_1 \begin{bmatrix} \mathbf{I}_n \\ \mathbf{P} \end{bmatrix} \mathbf{P}$$

  where

$$\Phi_0 = \begin{bmatrix} \mathbf{0}_{n\times n} & \mathbf{I}_n \\ -\mathbf{C} & -\mathbf{B} \end{bmatrix} \qquad \text{and} \qquad \Phi_1 = \begin{bmatrix} \mathbf{I}_n & \mathbf{0}_{n\times n} \\ \mathbf{0}_{n\times n} & \mathbf{A} \end{bmatrix}$$

- Before we can take this any further, we'll need some new tools: the **QZ Decompositions** and **Generalized Eigenvalues**

# QZ DECOMPOSITIONS

**DEFINITION 1**

The **QZ decomposition** of the $n \times n$ matrix pair $A$, $B$ is consists of the matrices $\mathbf{Q}$, $\mathbf{Z}$, $\mathbf{S}$, and $\mathbf{T}$ such that $A = \mathbf{QSZ}^*$, $B = \mathbf{QTZ}^*$, and the following statements are true:

1. $\mathbf{S}$ and $\mathbf{T}$ are upper triangular;
2. $\mathbf{Q}$ and $\mathbf{Z}$ are unitary, which means $\mathbf{QQ}^* = \mathbf{Q}^*\mathbf{Q} = \mathbf{I}_n$ and $\mathbf{ZZ}^* = \mathbf{Z}^*\mathbf{Z} = \mathbf{I}_n$, where $^*$ denotes the complex conjugate.

- A QZ decomposition is unique only up to the ordering of entries on the diagonals! The ratios of the diagonal entries $\left| T_{i,i}^{-1} S_{i,i} \right|$ are generally unique, however
- We can choose a specific QZ decomposition based on how we want to order the ratios

# GENERALIZED EIGENPROBLEM

**DEFINITION 2**

1. A **generalized eigenvalue** $\lambda$ of the $n \times n$ matrix pair $A, B$ is a value such that $\det(A - \lambda B) = 0$
2. A **generalized eigenvector** $v$ of the $n \times n$ matrix pair $A, B$ is an $n \times 1$ vector such that $Av = \lambda Bv$, where $\lambda$ is a generalized eigenvalue

- There is a special relationship between the generalized eigenvalues $\lambda_i$ of a matrix pair $A, B$ and the ratios of the diagonal entries of the corresponding QZ decompositions $T_{i,i}^{-1} S_{i,i}$ : *they're the same!* This will be a very useful fact...

# PUTTING IT TO WORK

- Define the QZ decomposition associated with $\Phi_0$ and $\Phi_1$ such that

$$\Phi_0 = \begin{bmatrix} \mathbf{Q}_{11} & \mathbf{Q}_{12} \\ \mathbf{Q}_{21} & \mathbf{Q}_{22} \end{bmatrix} \begin{bmatrix} \mathbf{S}_{11} & \mathbf{S}_{12} \\ \mathbf{0}_{n\times n} & \mathbf{S}_{22} \end{bmatrix} \begin{bmatrix} \mathbf{Z}_{11} & \mathbf{Z}_{12} \\ \mathbf{Z}_{21} & \mathbf{Z}_{22} \end{bmatrix}^*$$

$$\Phi_1 = \begin{bmatrix} \mathbf{Q}_{11} & \mathbf{Q}_{12} \\ \mathbf{Q}_{21} & \mathbf{Q}_{22} \end{bmatrix} \begin{bmatrix} \mathbf{T}_{11} & \mathbf{T}_{12} \\ \mathbf{0}_{n\times n} & \mathbf{T}_{22} \end{bmatrix} \begin{bmatrix} \mathbf{Z}_{11} & \mathbf{Z}_{12} \\ \mathbf{Z}_{21} & \mathbf{Z}_{22} \end{bmatrix}^*,$$

  where each block is $n \times n$, $\mathbf{S}_{ii}$ and $\mathbf{T}_{ii}$ are upper triangular, and we've chosen the QZ decomposition that sorts the $n$ smallest generalized eigenvalues into the top right corner
- Given this, it's possible to show (annoying, though) that $\mathbf{P} = \mathbf{Q}_{11}\mathbf{S}_{11}\mathbf{T}_{11}^{-1}\mathbf{Q}_{11}^{-1} = \mathbf{Z}_{21}\mathbf{Z}_{11}^{-1}$ satisfies the matrix quadratic in equation (3)

# THE SOLUTION AT LAST

- We now have expressions for the law of motion in terms of the model parameters:

$$\mathbf{P} = \mathbf{Q}_{11}\mathbf{S}_{11}\mathbf{T}_{11}^{-1}\mathbf{Q}_{11}^{-1} = \mathbf{Z}_{21}\mathbf{Z}_{11}^{-1} \tag{5}$$

$$\mathbf{R} = -\left(\mathbf{AP} + \mathbf{B}\right)^{-1}\mathbf{F} \tag{6}$$

**DEFINITION 3**

The law of motion $x_t = \mathbf{P}x_{t-1} + \mathbf{R}u_t$ constitutes a **linear rational expectations equilibrium** if it satisfies equation (1) and the boundary condition

$$\lim_{j\to\infty} \mathbb{E}_t\left[x_{t+j}\right] = \mathbf{0}_{n\times 1}. \tag{7}$$

# THE SOLUTION AT LAST

- We now have expressions for the law of motion in terms of the model parameters:

$$\mathbf{P} = \mathbf{Q}_{11}\mathbf{S}_{11}\mathbf{T}_{11}^{-1}\mathbf{Q}_{11}^{-1} = \mathbf{Z}_{21}\mathbf{Z}_{11}^{-1} \quad (5)$$

$$\mathbf{R} = -\left(\mathbf{A}\mathbf{P} + \mathbf{B}\right)^{-1}\mathbf{F} \quad (6)$$

- **Existence**: Can we find a stable **P**?
- **Uniqueness**: How many stable **P**s are there?
- How do we know that solution *exists*? How do we know it's *unique*?

# EXISTENCE AND UNIQUENESS

- **A Very Important Fact**: Recall that the diagonal of $S_{11}T_{11}^{-1}$ contains $n$ generalized eigenvalues and *we can choose* which $n$ of the $2n$ generalized eigenvalues are on the diagonal (since the QZ decomposition is non-unique)!

**CONDITIONS FOR EXISTENCE AND UNIQUENESS**

A *generalized eigenvalue* $\lambda$ of our system solves

$$\det(\Phi_0 - \lambda\Phi_1) = 0.$$

1. A rational expectations equilibrium **exists** if there are *at least* $n$ generalized eigenvalues less than 1 in absolute value;
2. A rational expectations equilibrium is **unique** if there are *exactly* n generalized eigenvalues less than 1 in absolute value.

# GENSYS

- Gensys starts with a linearized model, but in a form that's slightly different from equation (1):

$$\mathbf{A}x_t = \mathbf{B}x_{t-1} + \mathbf{C}\eta_t + \mathbf{F}u_t, \quad (8)$$

  where **A** might be singular, so we can't just invert it directly
- In this equation, $\eta_t$ is a $p \times 1$ vector of *expectational errors*, the difference between the a variable as it is realized at time $t+1$ and its expectation at time $t$
- While we impose distributional assumptions on $u_t$ (same ones as before), there are *no restrictions* on $\eta_t$ beyond $\mathbb{E}_t[\eta_t] = \mathbf{0}_{p\times 1}$ for all $t$
- The essence of this method is *using the boundary condition* to pin down $\eta_t$'s behavior

# RETURN OF THE QZ DECOMPOSITION

- We start by taking the QZ decomposition of the matrix pair $\mathbf{A}, \mathbf{B}$ such that $\mathbf{A} = \mathbf{QSZ}^*$ and $\mathbf{B} = \mathbf{QTZ}^*$. We also define $w_t = \mathbf{Z}^* x_t$, so we have

$$\mathbf{S} w_t = \mathbf{T} w_{t-1} + \underbrace{\mathbf{Q}^*\mathbf{C}}_{\widetilde{\mathbf{C}}} \eta_t + \underbrace{\mathbf{Q}^*\mathbf{F}}_{\widetilde{\mathbf{F}}} u_t$$

- We choose the QZ decomposition so that the entries on the diagonal of $\mathbf{S}^{-1}\mathbf{T}$ that are greater than 1 are in the lower right corner so

$$\begin{bmatrix} \mathbf{S}_{ss} & \mathbf{S}_{su} \\ \mathbf{0}_{n_u \times n_s} & \mathbf{S}_{uu} \end{bmatrix} \begin{bmatrix} w_{s,t} \\ w_{u,t} \end{bmatrix} = \begin{bmatrix} \mathbf{T}_{ss} & \mathbf{T}_{su} \\ \mathbf{0}_{n_u \times n_s} & \mathbf{T}_{uu} \end{bmatrix} \begin{bmatrix} w_{s,t-1} \\ w_{u,t-1} \end{bmatrix} + \begin{bmatrix} \widetilde{\mathbf{C}}_s \\ \widetilde{\mathbf{C}}_u \end{bmatrix} \eta_t + \begin{bmatrix} \widetilde{\mathbf{F}}_s \\ \widetilde{\mathbf{F}}_u \end{bmatrix} u_t.$$

# THE EXISTENCE CONDITION

- Because we require the boundary condition, equation (7), to hold, we need $w_{u,t} = w_{u,t-1} = \mathbf{0}_{n_u \times 1}$. Otherwise, the unstable coefficients $\mathbf{S}_{uu}^{-1}\mathbf{T}_{uu}$ would cause the system to explode. We therefore need $\widetilde{\mathbf{C}}_u \eta_t + \widetilde{\mathbf{F}}_u u_t = \mathbf{0}_{n_u \times 1}$ for all $t$, which implies our existence condition

**CONDITION FOR EXISTENCE**

In order for a rational expectations equilibrium to **exist**, the column space of $\widetilde{\mathbf{F}}_u$ must be contained within the column space of $\widetilde{\mathbf{C}}_u$. That is,

$$\text{span}\left(\widetilde{\mathbf{F}}_u\right) \subset \text{span}\left(\widetilde{\mathbf{C}}_u\right). \tag{9}$$

# BUT, HOW DOES IT WORK?

- Okay, fine. Suppose that $\text{span}\left(\widetilde{\mathbf{F}}_u\right) \not\subset \text{span}\left(\widetilde{\mathbf{C}}_u\right)$
- For any $u_t$, $-\widetilde{\mathbf{F}}_u u_t \in \text{span}\left(\widetilde{\mathbf{F}}_u\right)$. If $\text{span}\left(\widetilde{\mathbf{F}}_u\right) \not\subset \text{span}\left(\widetilde{\mathbf{C}}_u\right)$, then there exists some $u_t$ such that $-\widetilde{\mathbf{F}}_u u_t \notin \text{span}\left(\widetilde{\mathbf{C}}_u\right)$, which means there **cannot exist** any $\eta_t$ such that $-\widetilde{\mathbf{F}}_u u_t = \widetilde{\mathbf{C}}_u \eta_t$. As a consequence, there are realizations of $u_t$ such that $\widetilde{\mathbf{C}}_u \eta_t + \widetilde{\mathbf{F}}_u u_t \neq \mathbf{0}_{n_u \times 1}$, which can never happen if we have a rational expectations equilibrium

# THE UNIQUENESS CONDITION

- Suppose an *equilibrium exists*. Then

$$w_{s,t} = \mathbf{S}_{ss}^{-1}\mathbf{T}_{ss}w_{s,t-1} + \mathbf{S}_{ss}^{-1}\widetilde{\mathbf{C}}_s\eta_t + \mathbf{S}_{ss}^{-1}\widetilde{\mathbf{F}}_s u_t$$

and our equilibrium is only unique if there is a unique matrix $\Omega$ such that $\widetilde{\mathbf{C}}_s = \Omega\widetilde{\mathbf{C}}_u$

**CONDITION FOR UNIQUENESS**

In order for a rational expectations equilibrium to be **unique**, the row space of $\widetilde{\mathbf{C}}_s$ must be contained within the row space of $\widetilde{\mathbf{C}}_u$. That is,

$$\text{span}\left(\widetilde{\mathbf{C}}_s'\right) \subset \text{span}\left(\widetilde{\mathbf{C}}_u'\right). \tag{10}$$

# CHECKING EXISTENCE AND UNIQUENESS

- The ***existence condition*** implies there exists $\Gamma$ such that $\widetilde{\mathbf{F}}_u = \widetilde{\mathbf{C}}_u \Gamma$
- Let $\widetilde{\mathbf{C}}_u$ have SVD $UDV'$. Then existence requires

$$\mathbf{0} = \left(\widetilde{\mathbf{C}}_u V D^{-1} U' - \mathbf{I}_n\right) \widetilde{\mathbf{F}}_u,$$

  which we can check
- Similarly, uniqueness requires

$$\mathbf{0} = \widetilde{\mathbf{C}}_s \left(V D^{-1} U' \widetilde{\mathbf{C}}_u - \mathbf{I}_n\right),$$

  which may also be verified. This also yields our crucial $\Omega$ matrix

$$\Omega = \widetilde{\mathbf{C}}_s V D^{-1} U' \tag{11}$$

# THE SOLUTION

- Now, suppose we have an $\Omega$ and it's unique, then we obtain the rational expectations solution $x_t = \mathbf{P} x_{t-1} + \mathbf{Q} u_t$, where

$$\mathbf{P} = \mathbf{Z} \begin{bmatrix} \mathbf{S}_{ss}^{-1}\mathbf{T}_{ss} & \mathbf{0}_{n_s \times n_u} \\ \mathbf{0}_{n_u \times n_s} & \mathbf{0}_{n_u \times n_u} \end{bmatrix} \mathbf{Z}^* \tag{12}$$

$$\mathbf{Q} = \mathbf{Z} \begin{bmatrix} \mathbf{S}_{ss}^{-1}(\widetilde{\mathbf{F}}_s - \Omega \widetilde{\mathbf{F}}_u) \\ \mathbf{0}_{n_u \times m} \end{bmatrix} \tag{13}$$

- This is everything Dynare is doing when it solves a model you throw into it!

# THE END

- We've covered two common methods for solving linear rational expectations models
- There are others, but these are the most widely-used approaches
- The ideas behind the two are pretty similar and, despite the tons of matrix algebra, pretty simple
- *Theoretically*, the QZ decomposition isn't necessary to deal with singularity or anything else we're working with. However, *in practice*, **we need it** because it does the same things in a **numerically stable** way
- Pick your favorite! They'll get you the same thing, anyway