Responses from sequences of expectations. The fake news matrix $\mathcal{F}$ maps the expectations (of current and future variables) in a single period to outcomes in current and future periods. Maybe that's unclear, so I'll be more concrete.

Let $\mathcal{Z}_{u,v} = \mathbb{E}_u[Z_v]$ be the expectation of Z_v (the value of Z in period v) in period u. What is the effect of a change $d\mathcal{Z}_{u,v}$ on an outcome Y_t? That would be $dY_{t,u,v} = \mathcal{F}_{t-u,v-u} d\mathcal{Z}_{u,v}$. Why? $\mathcal{F}_{i,j}$ maps the change in expectations j periods in the future to an outcome i periods in the future. In period u, v is $v - u$ periods in the future and t is $t - u$ periods in the future.

Of course, $dY_{t,u,v}$ only captures a partial effect, the effect of decisions made in period u on Y_t due to beliefs about period v in period u. If I want to capture the full effect of changing beliefs about period v on Y_t (call that $dY_{t,v}$), I must include the effects of decisions in other periods. Doing so implies

$$dY_{t,v} = \sum_{u=0}^{\min\{v,t\}} dY_{t,u,v} = \sum_{u=0}^{\min\{v,t\}} \mathcal{F}_{t-u,v-u} d\mathcal{Z}_{u,v}. \tag{1}$$

Why do I only go up to $\min\{v, t\}$? If $u > t$, Y_t has already been realized and nothing believed or done in period u can change that. If $u > v$, the shock has already come and gone so changing expectations has no effect.

From expectations to reality. The problem now is that $dY_{t,v}$ is solely a function of sequences of expectations. I'd like to know how Y_t is affected by changes in actual variables. That means I need a mapping $E_{u,v,s}$ from an actual variable in period s dX_s to $d\mathcal{Z}_{u,v}$, so that

$$d\mathcal{Z}_{u,v} = \sum_{s=0}^{\infty} E_{u,v,s} dX_s. \tag{2}$$

Plugging (2) into (1), I obtain

$$dY_{t,v} = \sum_{s=0}^{\infty} \sum_{u=0}^{\min\{v,t\}} \mathcal{F}_{t-u,v-u} E_{u,v,s} dX_s, \tag{3}$$

which I can sum over v to obtain the total change in dY_t due to the sequence of actual changes $d\mathbf{X}$. This is

$$dY_t = \sum_{v=0}^{\infty} dY_{t,v} = \sum_{v=0}^{\infty}\sum_{s=0}^{\infty}\sum_{u=0}^{\min\{v,t\}} \mathcal{F}_{t-u,v-u}E_{u,v,s}dX_s. \tag{4}$$

Equation (4) allows me to ask how dX_s *alone* affects dY_t. This is simply

$$dY_{t,s} = \sum_{v=0}^{\infty}\sum_{u=0}^{\min\{v,t\}} \mathcal{F}_{t-u,v-u}E_{u,v,s}dX_s \tag{5}$$

and entry (t,s) of the Jacobian $\mathcal{J}^{Y,X}$ mapping $d\mathbf{X}$ to $d\mathbf{Y}$ is therefore

$$\mathcal{J}_{t,s} = \sum_{v=0}^{\infty}\sum_{u=0}^{\min\{v,t\}} \mathcal{F}_{t-u,v-u}E_{u,v,s}. \tag{6}$$

What is $E_{u,v,s}$? This general mapping E can capture a lot of things, as it's simply the derivative

$$E_{u,v,s} = \frac{d\mathbb{E}_u[Z_v]}{dX_s}. \tag{7}$$

This makes it exceedingly easy to take a model of expectation formation and throw it into sequence-space.

Under rational expectations, $E_{u,v,s} = \mathbb{I}(v = s)$ if $Z = X$ and 0 otherwise. In this case, (6) yields the standard fake news cumulation

$$\mathcal{J}_{t,s} = \sum_{u=0}^{\min\{s,t\}} \mathcal{F}_{t-u,s-u}. \tag{8}$$

Under myopia, $E_{u,v,s} = \theta^{\min\{v-u,0\}}\mathbb{I}(v = s)$,[1] which yields

$$\mathcal{J}_{t,s} = \sum_{u=0}^{\min\{s,t\}} \theta^{\min\{s-u,0\}}\mathcal{F}_{t-u,s-u}. \tag{9}$$

[1] Again, if $Z = X$ and 0 otherwise. The same caveat will apply for sticky information.

Under sticky expectations,[2] $E_{u,v,s} = (1 - \mathbb{I}(u < v)\theta^u)\mathbb{I}(v = s)$, which yields

$$\mathcal{J}_{t,s} = \sum_{u=0}^{\min\{s,t\}} (1 - \mathbb{I}(u < s)\theta^u)\mathcal{F}_{t-u,s-u}. \tag{10}$$

Obviously then, if there's time-separability (i.e. $E_{u,v,s} = \widetilde{E}_{u,s}\mathbb{I}(v = s)$, meaning variables in a period do not affect expectations in other periods), the cumulation becomes

$$\mathcal{J}_{t,s} = \sum_{u=0}^{\min\{s,t\}} \widetilde{E}_{u,s}\mathcal{F}_{t-u,s-u}. \tag{11}$$

Models of expectation formation that do not satisfy the formulation in (11) include, for example, adaptive expectations and models of learning. However, they can be easily accommodated by the formulation in (6).

Jacobians in models with non-separability. In a model with some forecasting rule with perfect observation of variables after they're realized

$$E_u[Z_v] = Z_v + \mathbb{I}(u < v)\left(\sum_{j=0}^{p} \beta_{v-u,-j}Z_{u-j} - Z_v\right), \tag{12}$$

the expectation matrix is

$$E_{u,v,s} = (1 - \mathbb{I}(u < v))\mathbb{I}(v = s) + \mathbb{I}(u < v)\mathbb{I}(s \le u \le s + p)\beta_{v-u,s-u}. \tag{13}$$

[2] In this formulation, assume that everyone learns the value of dZ_v when it hits, so there's no distortion in periods $u \ge v$.